Global energy minimization for multiple fracture growth

Danas Sutula

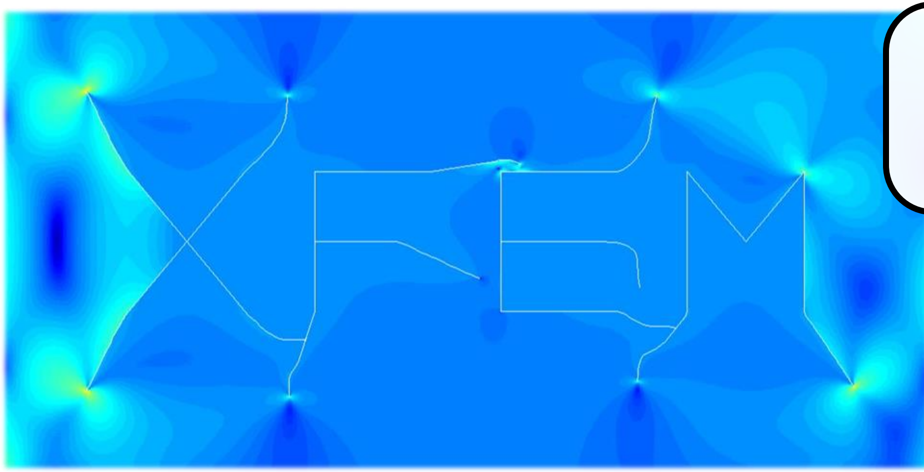
Prof. Stephane Bordas & Dr. Pierre Kerfriden

(10/12/13)

Discretization: XFEM

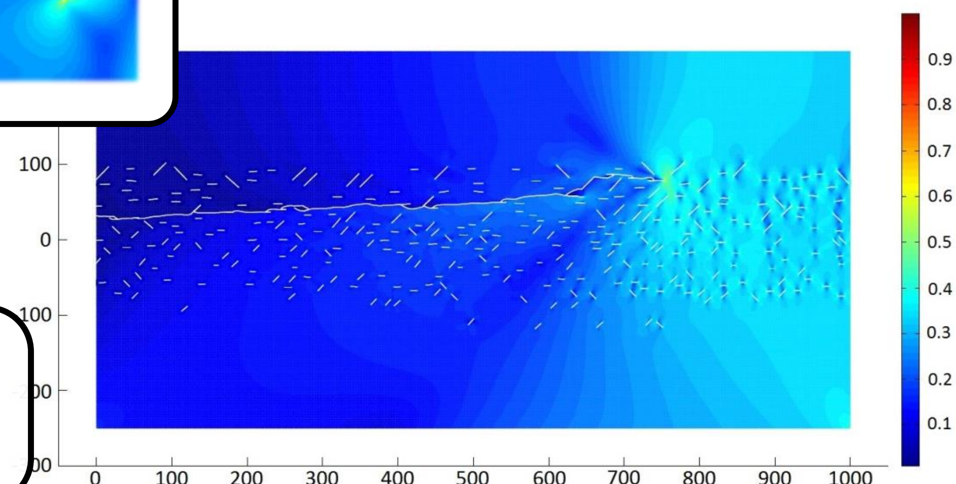
Extended Finite Element Method (XFEM)

- Introduced by Ted Belytschko et al. (1999) for elastic problems



Fracture of "XFEM" using XFEM

- pressure driven crack propagation



Fracture process (>300 cracks)

- vertical tensile loading

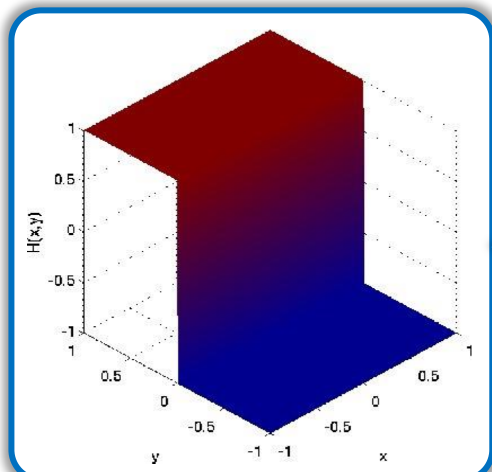
Discretization: XFEM

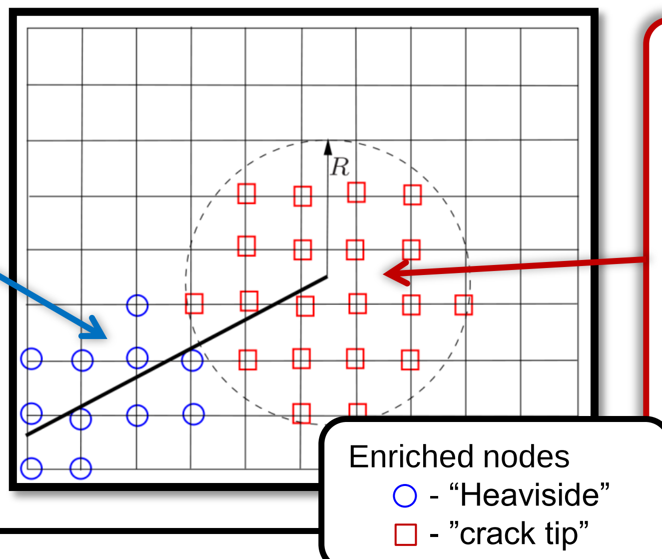
Approximation function :

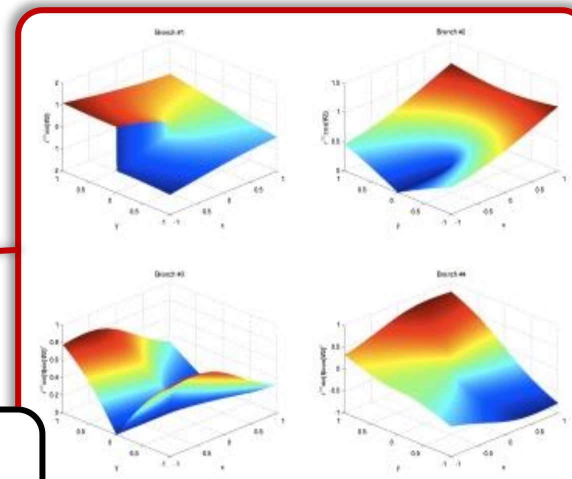
$$\mathbf{u}^h(\mathbf{x}) = \underbrace{\sum_{I \in \mathcal{N}_I} N_I(\mathbf{x})\mathbf{u}^I}_{\text{standard part}} + \underbrace{\sum_{J \in \mathcal{N}_J} N_J(\mathbf{x}) H(\mathbf{x}) \mathbf{a}^J}_{\text{discontinuous enrichment}} + \underbrace{\sum_{K \in \mathcal{N}_K} N_K(\mathbf{x}) \sum_{\alpha=1}^{4} f_\alpha(\mathbf{x}) \mathbf{b}^{K\alpha}}_{\text{singular tip enrichment}}$$

$$H(\mathbf{x}) = \begin{cases} +1 & \text{if } \mathbf{x} \text{ above crack} \\ -1 & \text{if } \mathbf{x} \text{ below crack} \end{cases}$$

$$\{f_\alpha(r,\theta), \alpha = 1,4\} = \left\{ \sqrt{r}\sin\frac{\theta}{2}, \sqrt{r}\cos\frac{\theta}{2}, \sqrt{r}\sin\frac{\theta}{2}\sin\theta, \sqrt{r}\cos\frac{\theta}{2}\sin\theta \right\}$$







Crack propagation: traditional approach

Evaluation of stress intensity factors (SIF)

- The interaction integral (Yau 1980)

(1) – from current solution
(2) – known auxiliary solution

$$I^{(1+2)} = \int_\Omega \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1+2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} \mathrm{d}\Omega = \frac{2}{E'} \left(K_I^{(1)} K_I^{(2)} + K_{II}^{(1)} K_{II}^{(2)} \right)$$

Crack growth criterion for mixed mode fracture

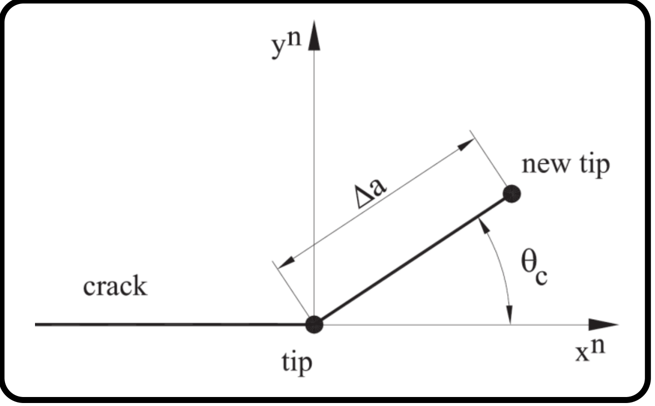
- Direction that maximises the energy release (Nuismer 1975)

$$\frac{k_I^2(K_I, K_{II}, \theta_{\text{inc}}) + k_{II}^2(K_I, K_{II}, \theta_{\text{inc}})}{E'} = G_c$$

Crack growth direction

- orthogonal to maximum hoop stress

$$\theta_c(K_I, K_{II}) = 2 \tan^{-1} \left[\frac{1}{4} \left(\frac{K_I}{K_{II}} - \text{sign}(K_{II}) \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \right) \right]$$



Crack propagation: optimization of directions

- Energy release rate w.r.t crack increment direction:

$$Gs_i = -\frac{\partial \Pi}{\partial \theta_i}$$

- The rates of the energy release rate are given by:

$$Hs_{i,j} = \frac{\partial Gs_i}{\partial \theta_j} = -\frac{\partial^2 \Pi}{\partial \theta_i \partial \theta_j}$$

- where, in a discrete setting, the potential energy is:

$$\Pi = \frac{1}{2} u' K u - u' f$$

Crack propagation: optimization of directions

- The discrete potential energy:

$$\Pi = \frac{1}{2}u'Ku - u'f$$

- The discrete energy release rate:

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f - \delta_i u'(Ku - f)$$

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f \quad \text{(where: } \delta_i = \frac{\partial}{\partial\theta_i} \text{)}$$

- The rates of the energy release rate

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ij}^2 Ku - u'\delta_{ij}^2 f\right) - \delta_j u'\left(\delta_i Ku - \delta_i f\right) \quad \text{(where: } \delta_{ij} = \frac{\partial^2}{\partial\theta_i\theta_j} \text{)}$$

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ii}^2 Ku - u'\delta_{ii}^2 F\right) + (\delta_j Ku - \delta_j f)'K^{-1}(\delta_i Ku - \delta_i f)$$

Crack propagation: optimization of directions

- The discrete potential energy:

$$\Pi = \frac{1}{2}u'Ku - u'f$$

- The discrete energy release rate:

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f - \delta_i u'(\cancelto{0}{Ku - f})$$

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f \quad \text{(where: } \delta_i = \frac{\partial}{\partial\theta_i}\text{)}$$

- The rates of the energy release rate

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ij}^2 Ku - u'\delta_{ij}^2 f\right) - \delta_j u'\left(\delta_i Ku - \delta_i f\right) \quad \text{(where: } \delta_{ij} = \frac{\partial^2}{\partial\theta_i\theta_j}\text{)}$$

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ii}^2 Ku - u'\delta_{ii}^2 F\right) + (\delta_j Ku - \delta_j f)'K^{-1}(\delta_i Ku - \delta_i f)$$

Crack propagation: optimization of directions

- The discrete potential energy:

$$\Pi = \frac{1}{2}u'Ku - u'f$$

- The discrete energy release rate:

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f - \delta_i u'(\underbrace{Ku - f}_{\mathbf{0}})$$

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f \quad \text{(where: } \delta_i = \frac{\partial}{\partial\theta_i}\text{)}$$

- The rates of the energy release rate

$$\boxed{\delta u = -K^{-1}(\delta K u - \delta f)}$$

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ij}^2 Ku - u'\delta_{ij}^2 f\right) - \delta_j u'(\delta_i Ku - \delta_i f) \quad \text{(where: } \delta_{ij} = \frac{\partial^2}{\partial\theta_i\theta_j}\text{)}$$

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ii}^2 Ku - u'\delta_{ii}^2 F\right) + (\delta_j Ku - \delta_j f)'K^{-1}(\delta_i Ku - \delta_i f)$$

Crack propagation: optimization of directions

- The discrete potential energy:

$$\Pi = \frac{1}{2}u'Ku - u'f$$

- The discrete energy release rate:

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f - \delta_i u'\cancelto{0}{(Ku - f)}$$

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f \quad \text{(where: } \delta_i = \frac{\partial}{\partial\theta_i}\text{)}$$

- The rates of the energy release rate

$$\boxed{\delta u = -K^{-1}(\delta Ku - \delta f)}$$

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ij}^2 Ku - u'\delta_{ij}^2 f\right) - \delta_j u'(\delta_i Ku - \delta_i f) \quad \text{(where: } \delta_{ij} = \frac{\partial^2}{\partial\theta_i\theta_j}\text{)}$$

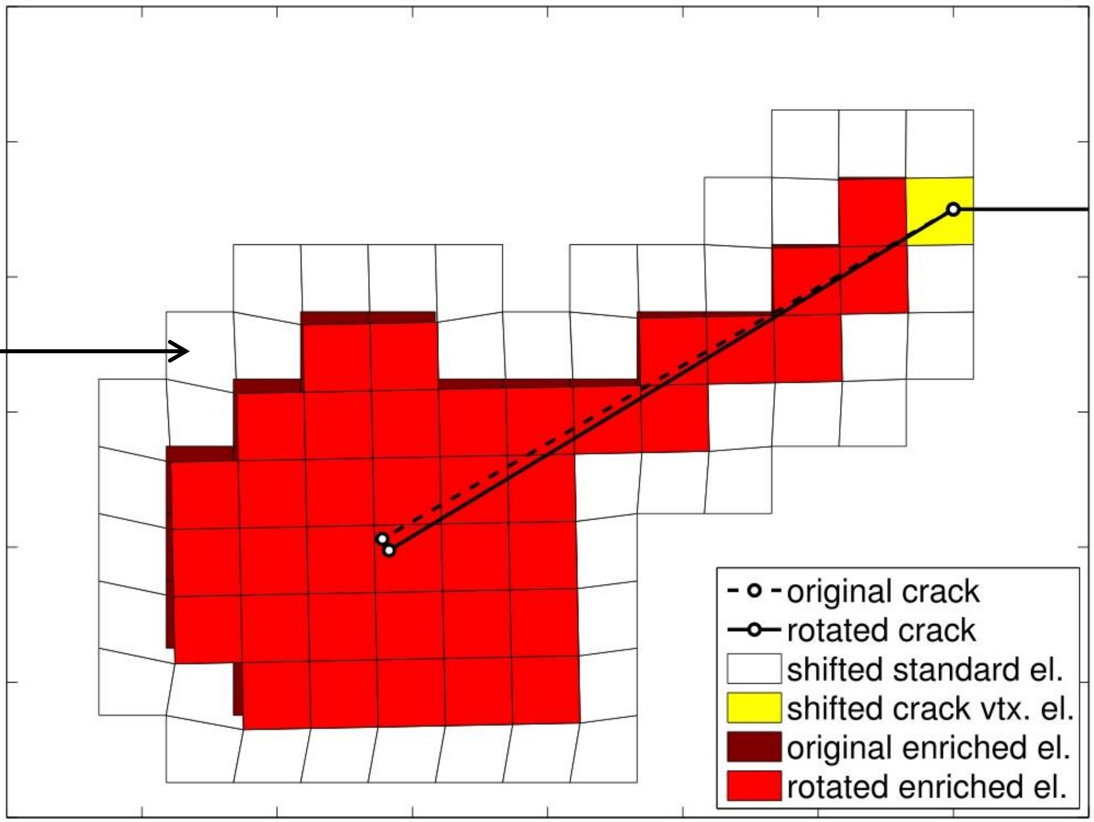
$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ii}^2 Ku - u'\delta_{ii}^2 F\right) + \underbrace{(\delta_j Ku - \delta_j f)'\overset{\text{expensive}}{\boxed{K^{-1}(\delta_i Ku - \delta_i f)}}}_{\text{remote interaction}}$$

Implementation

$$\delta\mathbf{K}_e = \int_{\Omega_e} \left(\delta\mathbf{B}^T\mathbf{D}\mathbf{B} + \mathbf{B}^T\mathbf{D}\delta\mathbf{B}\right) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\, \delta\det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

$$\delta^2\mathbf{K}_e = \int_{\Omega_e} \left(\delta^2\mathbf{B}^T\mathbf{D}\mathbf{B} + 2\delta\mathbf{B}^T\mathbf{D}\delta\mathbf{B} + \mathbf{B}^T\mathbf{D}\delta^2\mathbf{B}\right) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} 2\left(\delta\mathbf{B}^T\mathbf{D}\mathbf{B} + \mathbf{B}^T\mathbf{D}\delta\mathbf{B}\right) \delta\det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\, \delta^2\det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

Differentiation of the stiffness matrix
w.r.t. crack increment direction



Implementation

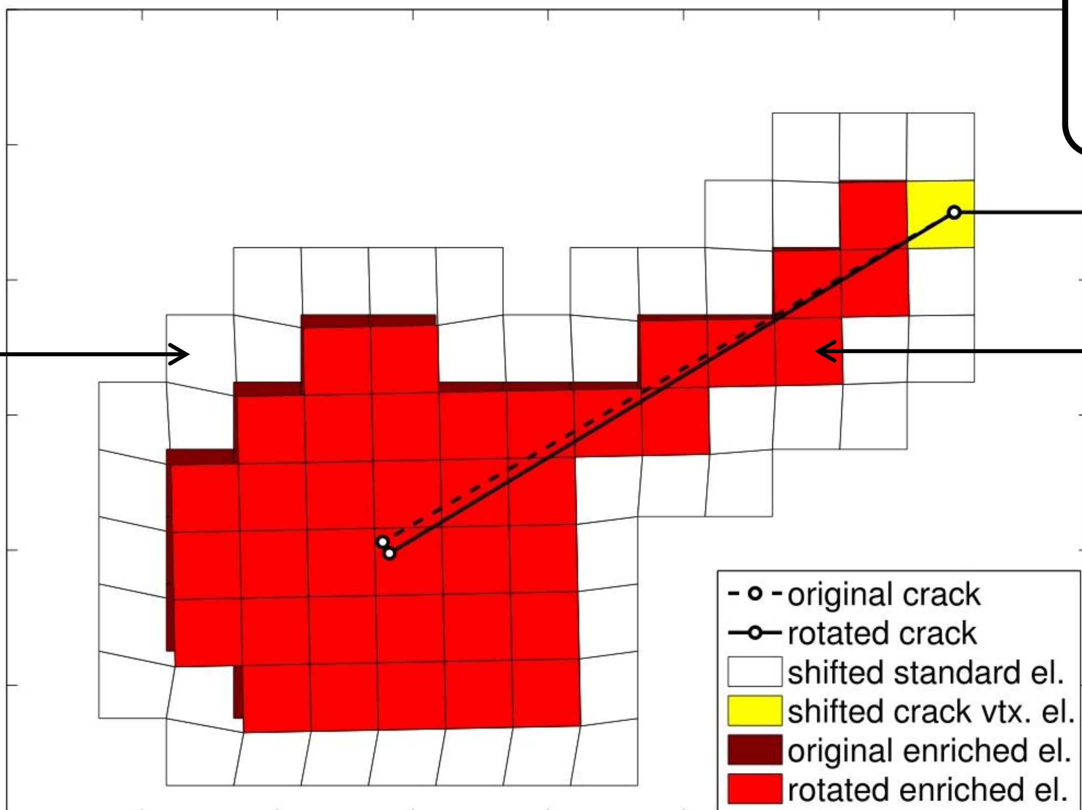
$$\delta \mathbf{K}_e = \int_{\Omega_e} \left(\delta \mathbf{B}^T \mathbf{D} \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B}\right) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, \delta \det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

$$\delta^2 \mathbf{K}_e = \int_{\Omega_e} \left(\delta^2 \mathbf{B}^T \mathbf{D} \mathbf{B} + 2\delta \mathbf{B}^T \mathbf{D} \delta \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta^2 \mathbf{B}\right) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} 2\left(\delta \mathbf{B}^T \mathbf{D} \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B}\right) \delta \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, \delta^2 \det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

Differentiation of the stiffness matrix w.r.t. crack increment direction

$$\delta \mathbf{K_e} = \mathbf{T}^T \mathbf{K_e} + \mathbf{K_e} \mathbf{T}$$

$$\delta^2 \mathbf{K_e} = 2(\mathbf{T}^T \mathbf{K_e} \mathbf{T} - \mathbf{K_e})$$



Implementation

- The discrete potential energy:

$$\Pi = \frac{1}{2}u'Ku - u'f$$

- The discrete energy release rate:

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f - \delta_i u' \overbrace{(Ku - f)}^{\mathbf{0}}$$

$$Gs_i = -\frac{1}{2}u'\delta_i Ku + u'\delta_i f \quad \text{(where: } \delta_i = \frac{\partial}{\partial\theta_i}\text{)}$$

- The rates of the energy release rate

$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ij}^2 Ku - u'\delta_{ij}^2 f\right) - \delta_j u'\left(\delta_i Ku - \delta_i f\right) \quad \text{(where: } \delta_{ij} = \frac{\partial^2}{\partial\theta_i\theta_j}\text{)}$$

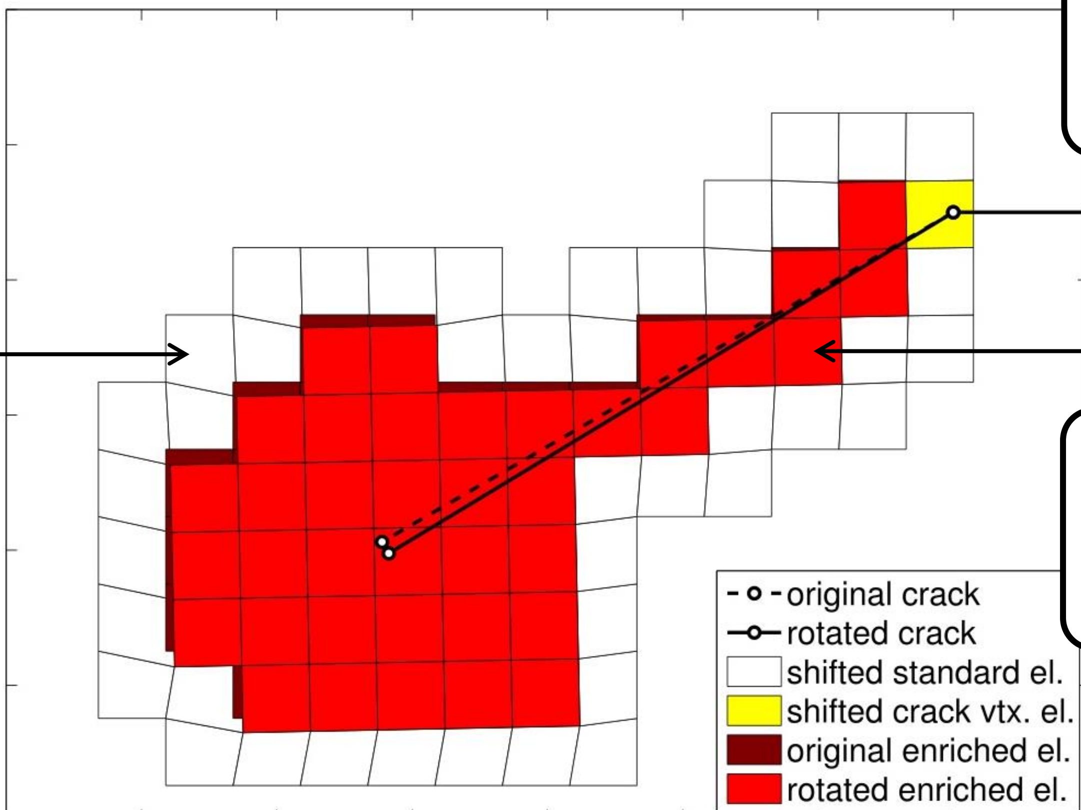
$$Hs_{ij} = -\left(\frac{1}{2}u'\delta_{ii}^2 Ku - u'\delta_{ii}^2 F\right) + (\delta_j Ku - \delta_j f)'K^{-1}(\delta_i Ku - \delta_i f)$$

Implementation

$$\delta \mathbf{K}_e = \int_{\Omega_e} (\delta \mathbf{B}^T \mathbf{D} \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B}) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, \delta \det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

$$\delta^2 \mathbf{K}_e = \int_{\Omega_e} (\delta^2 \mathbf{B}^T \mathbf{D} \mathbf{B} + 2\delta \mathbf{B}^T \mathbf{D} \delta \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta^2 \mathbf{B}) \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} 2\, (\delta \mathbf{B}^T \mathbf{D} \mathbf{B} + \mathbf{B}^T \mathbf{D} \delta \mathbf{B})\, \delta \det(\mathbf{J})\, \mathrm{d}\bar{\Omega} + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{B}\, \delta^2 \det(\mathbf{J})\, \mathrm{d}\bar{\Omega}$$

Differentiation of the stiffness matrix
w.r.t. crack increment direction



$$\delta \mathbf{K_e} = \mathbf{T}^T \mathbf{K_e} + \mathbf{K_e} \mathbf{T}$$

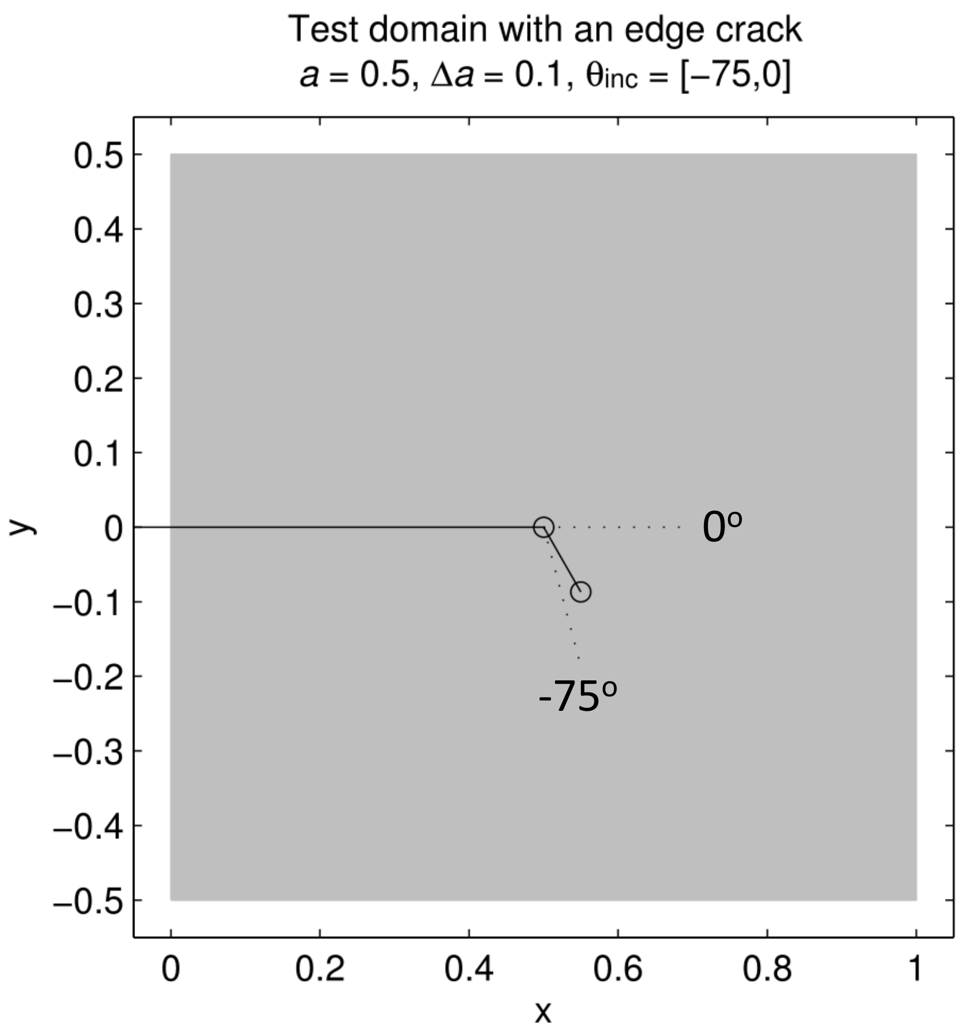
$$\delta^2 \mathbf{K_e} = 2(\mathbf{T}^T \mathbf{K_e} \mathbf{T} - \mathbf{K_e})$$

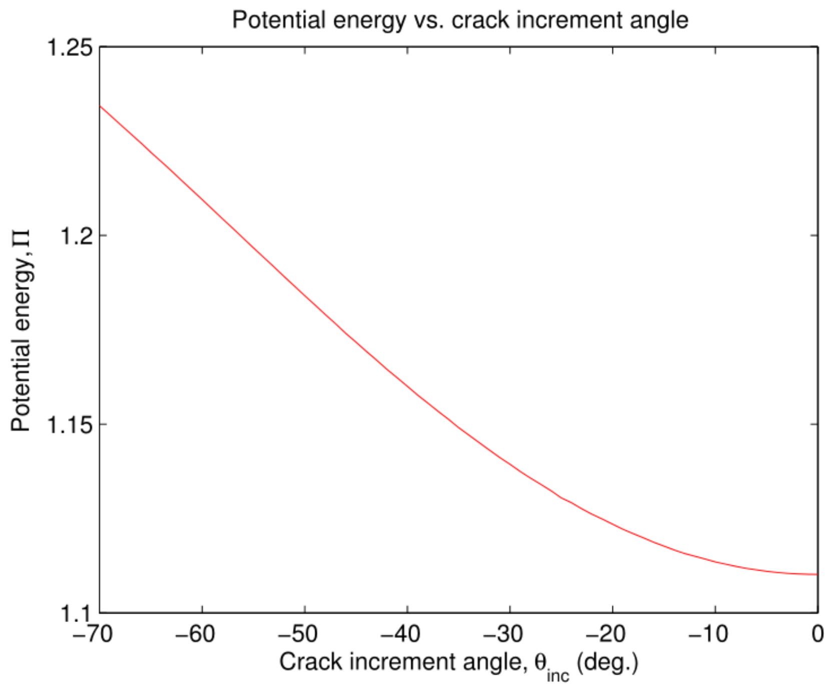
Updated directions:

$$\theta^{k+1} = \theta^k - \mathbf{Hs}^{-1} \mathbf{Gs}$$

Verification: energy release rates

- **Square plate in pure mode-I loading with varying increment angle**



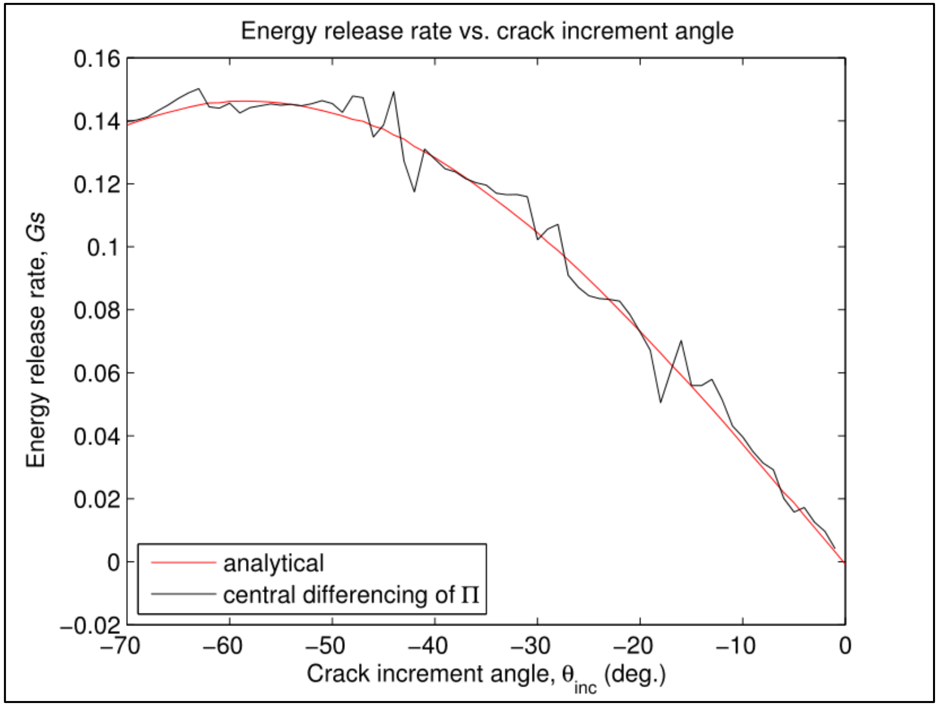


Potential energy vs. crack increment angle

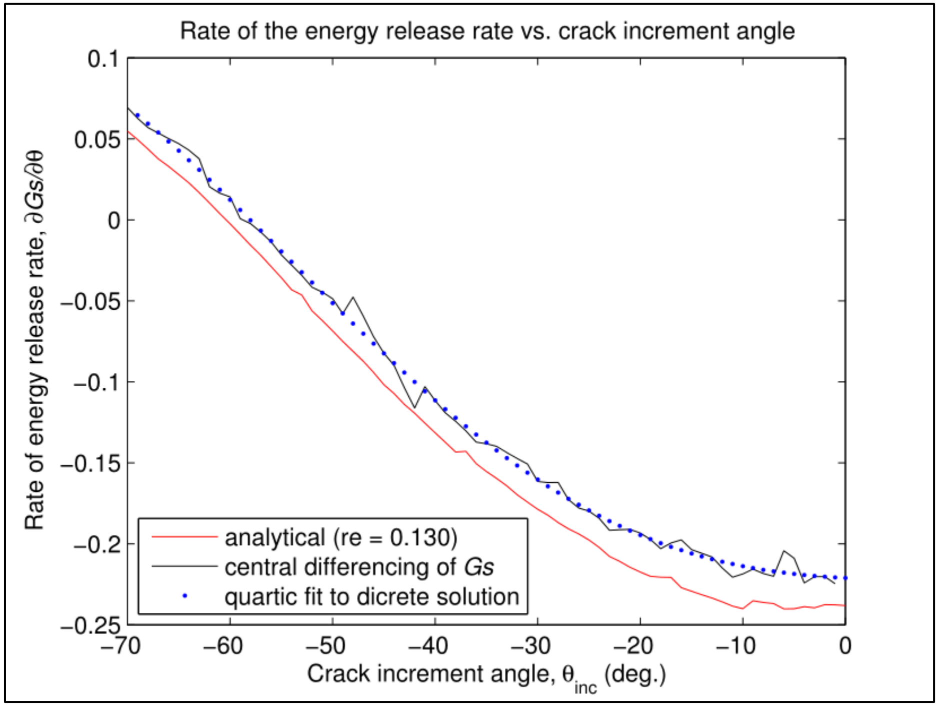
Verification: energy release rates

- **Square plate in pure mode-I loading with varying increment angle**

mesh= 50 x 50, $r_{tip} = 2.5h_e$, $\Delta a_{inc} = 0.1$



Energy release rate vs. crack increment angle

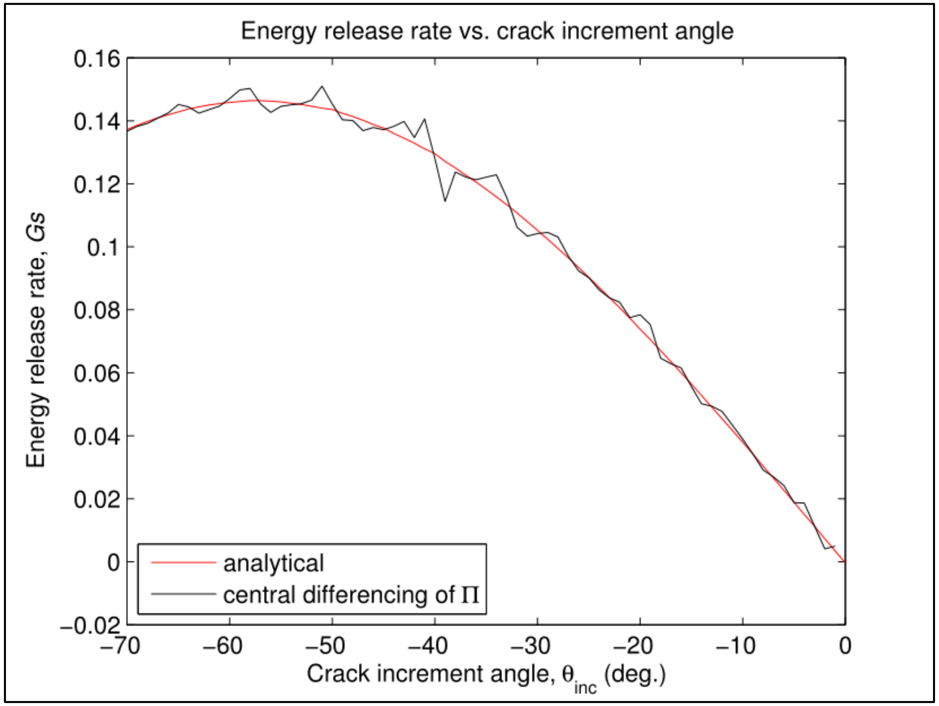


Rate of energy release vs. increment angle

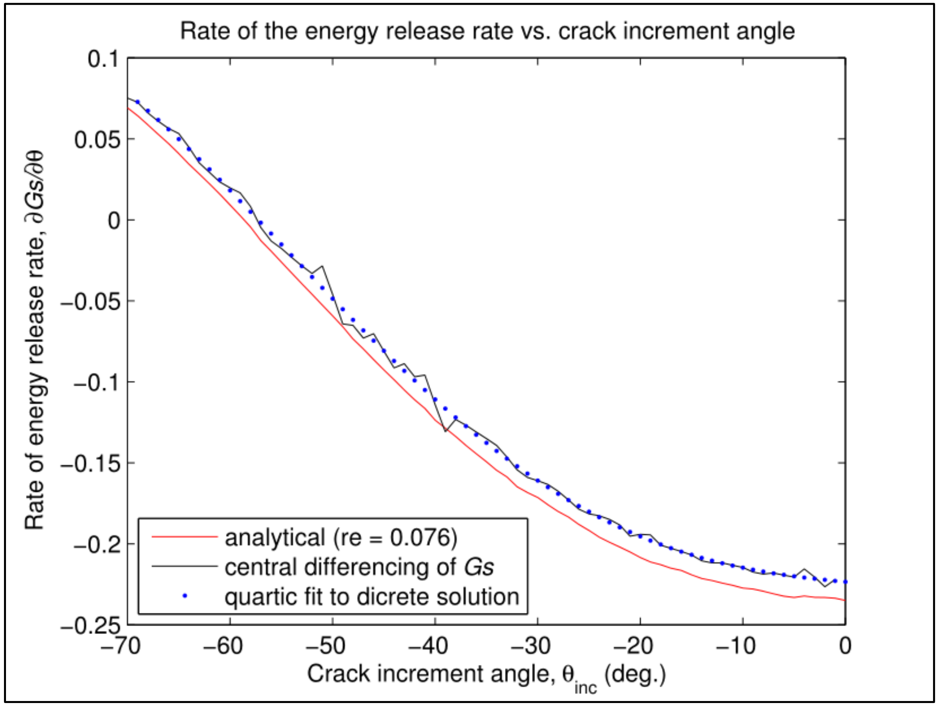
Verification: energy release rates

- **Square plate in pure mode-I loading with varying increment angle**

mesh= 100 x 100, $r_{tip} = 5.0h_e$, $\Delta a_{inc} = 0.1$



Energy release rate vs. crack increment angle

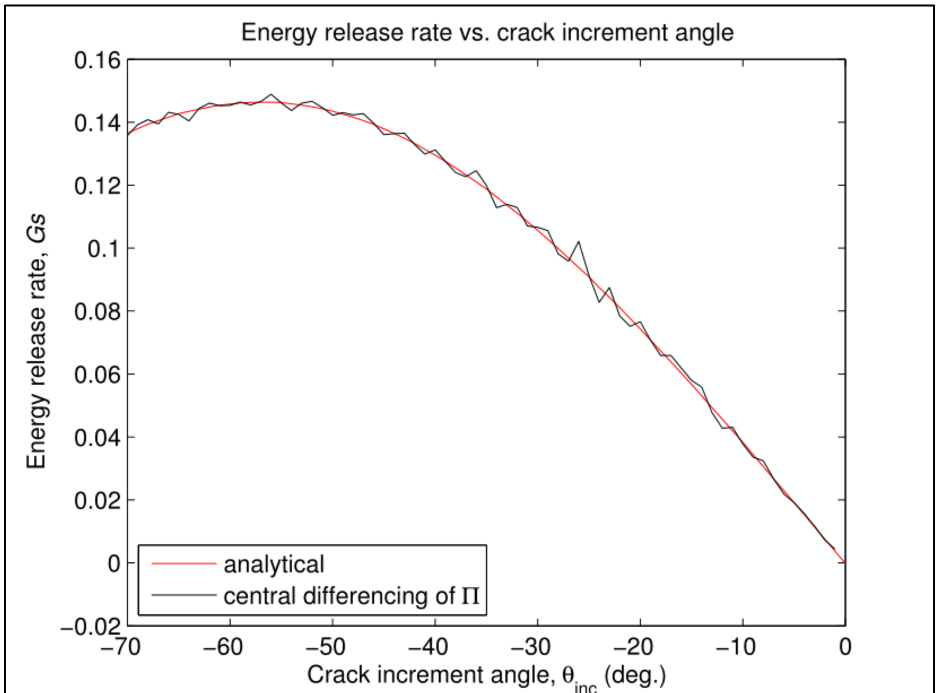


Rate of energy release vs. increment angle

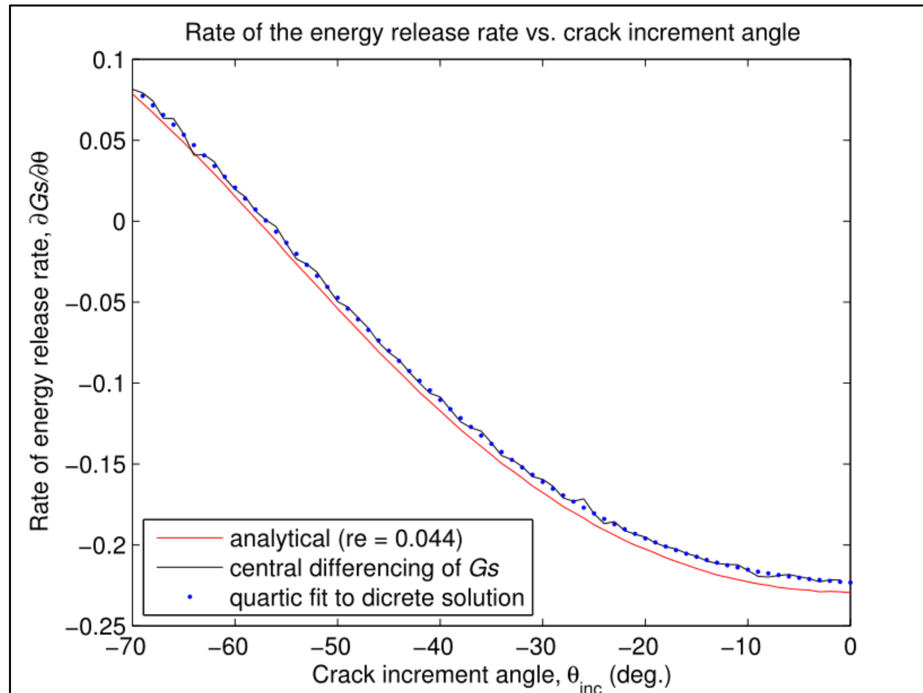
Verification: energy release rates

- **Square plate in pure mode-I loading with varying increment angle**

mesh= 200 x 200, $r_{tip} = 10.0h_e$, $\Delta a_{inc} = 0.1$



Energy release rate vs. crack increment angle



Rate of energy release vs. increment angle

Verification: energy release rates

- Geometrical tip enrichment ($\Delta a_{\mathrm{inc}} = 0.1$)

mesh = 50 x 50, $r_{tip} = 2.5h_e \longrightarrow \|re_{Hs}\|_{L2} = 0.130$

mesh = 100 x 100, $r_{tip} = 5.0h_e \longrightarrow \|re_{Hs}\|_{L2} = 0.076$

mesh = 200 x 200, $r_{tip} = 10.0h_e \longrightarrow \|re_{Hs}\|_{L2} = 0.044$

- Topological enrichment ($r_{\mathrm{tip}} = 2.5h_e$)

Verification: energy release rates

- Geometrical tip enrichment ($\Delta a_{\mathrm{inc}} = 0.1$)

mesh = 50 x 50, $r_{tip} = 2.5h_e \longrightarrow \|re_{Hs}\|_{L2} = 0.130$

mesh = 100 x 100, $r_{tip} = 5.0h_e \longrightarrow \|re_{Hs}\|_{L2} = 0.076$

mesh = 200 x 200, $r_{tip} = 10.0h_e \longrightarrow \|re_{Hs}\|_{L2} = 0.044$

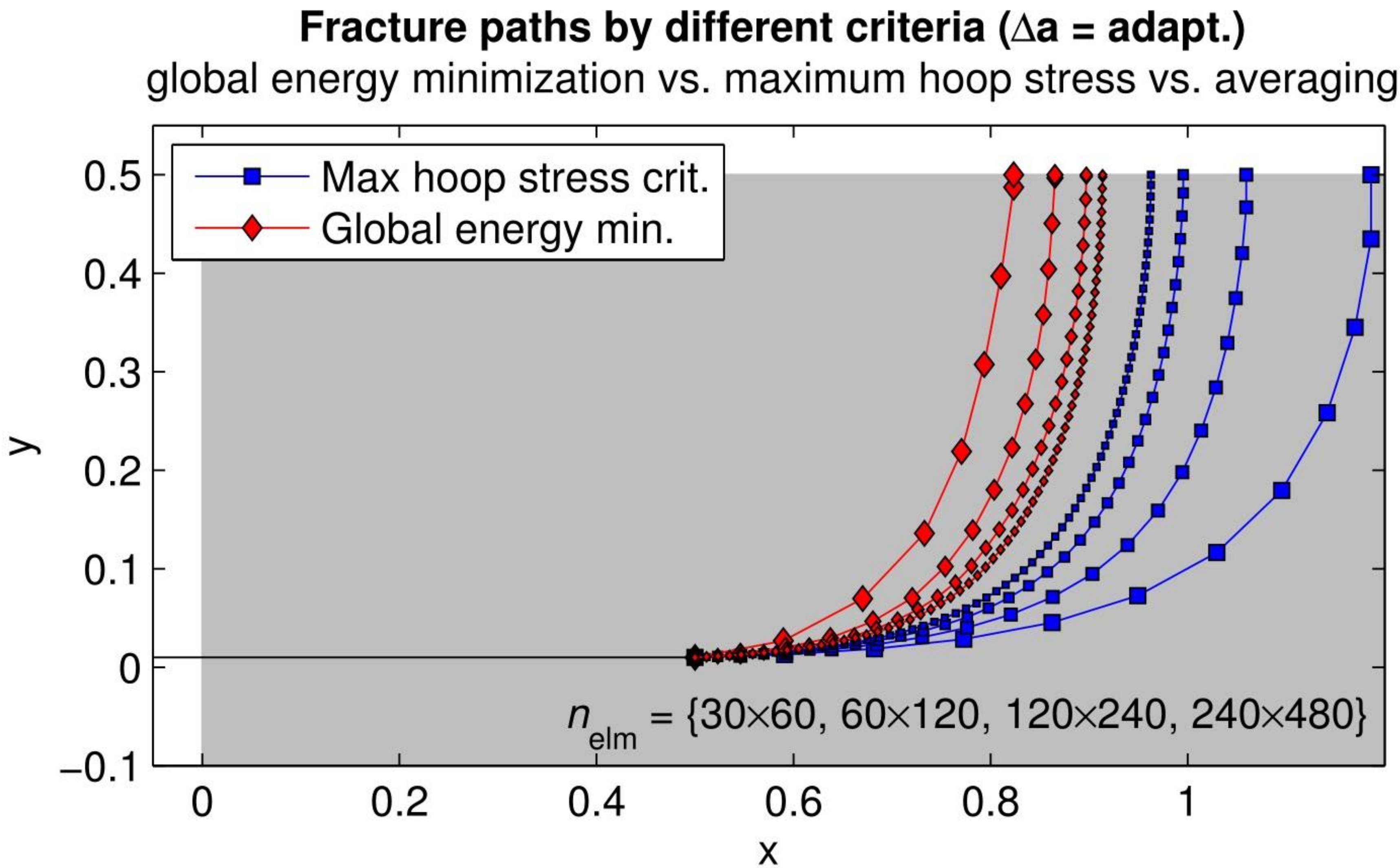
- Topological enrichment ($r_{\mathrm{tip}} = 2.5h_e$)

mesh = 50 x 50, $\Delta a_{inc} = 0.100 \longrightarrow \|re_{Hs}\|_{L2} = 0.130$

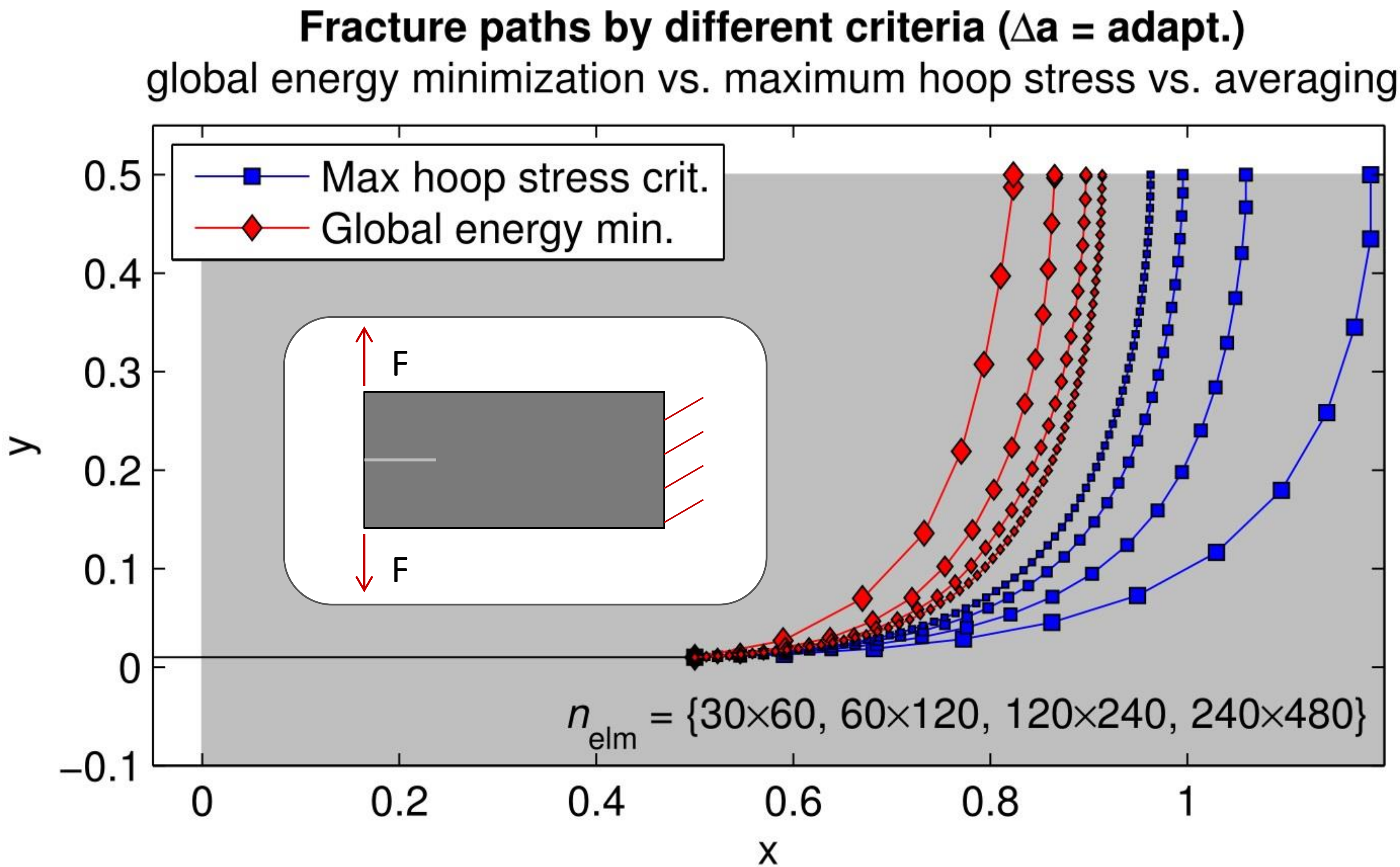
mesh = 100 x 100, $\Delta a_{inc} = 0.050 \longrightarrow \|re_{Hs}\|_{L2} = 0.134$

mesh = 200 x 200, $\Delta a_{inc} = 0.025 \longrightarrow \|re_{Hs}\|_{L2} = 0.139$

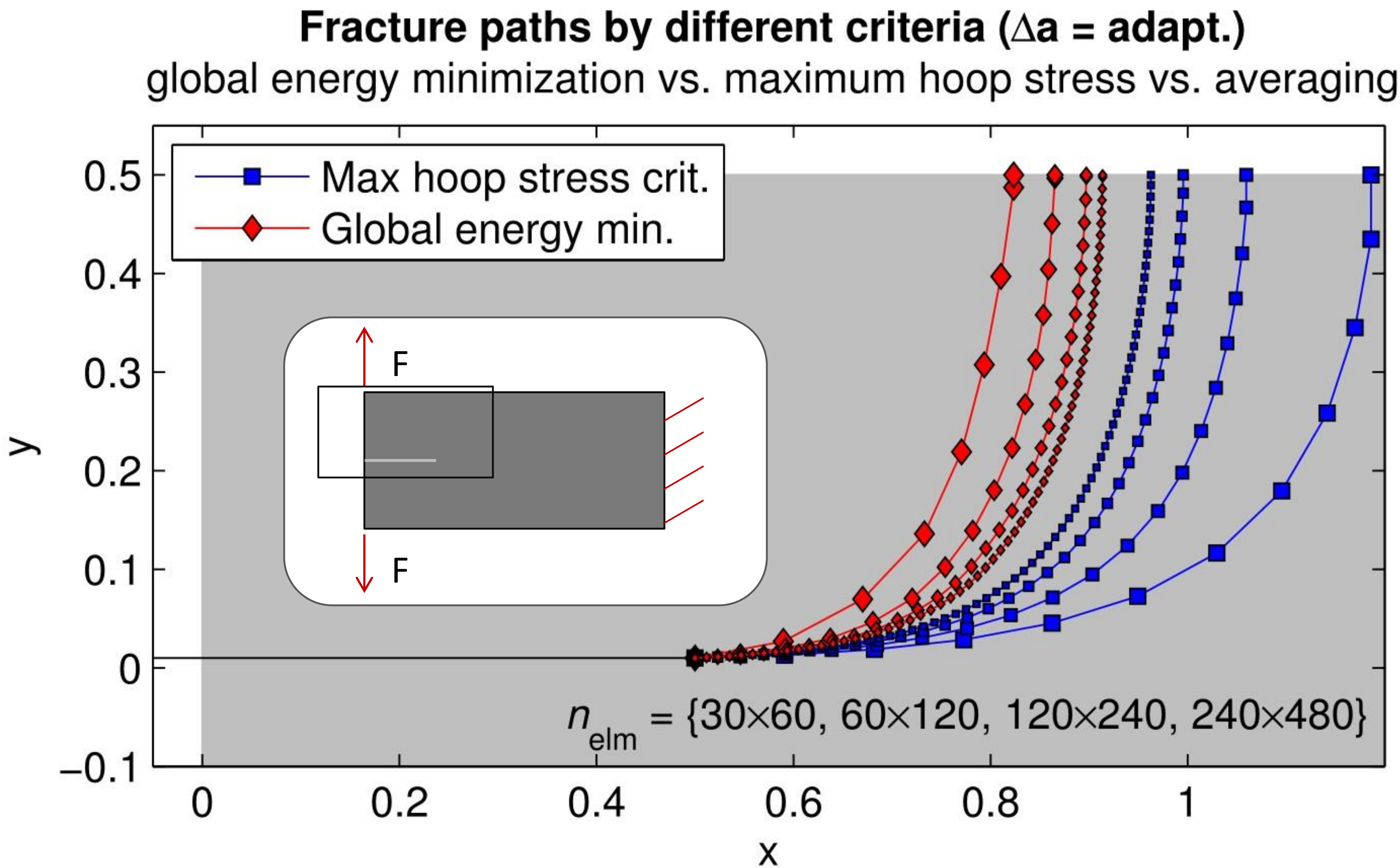
Results: Double cantilever problem



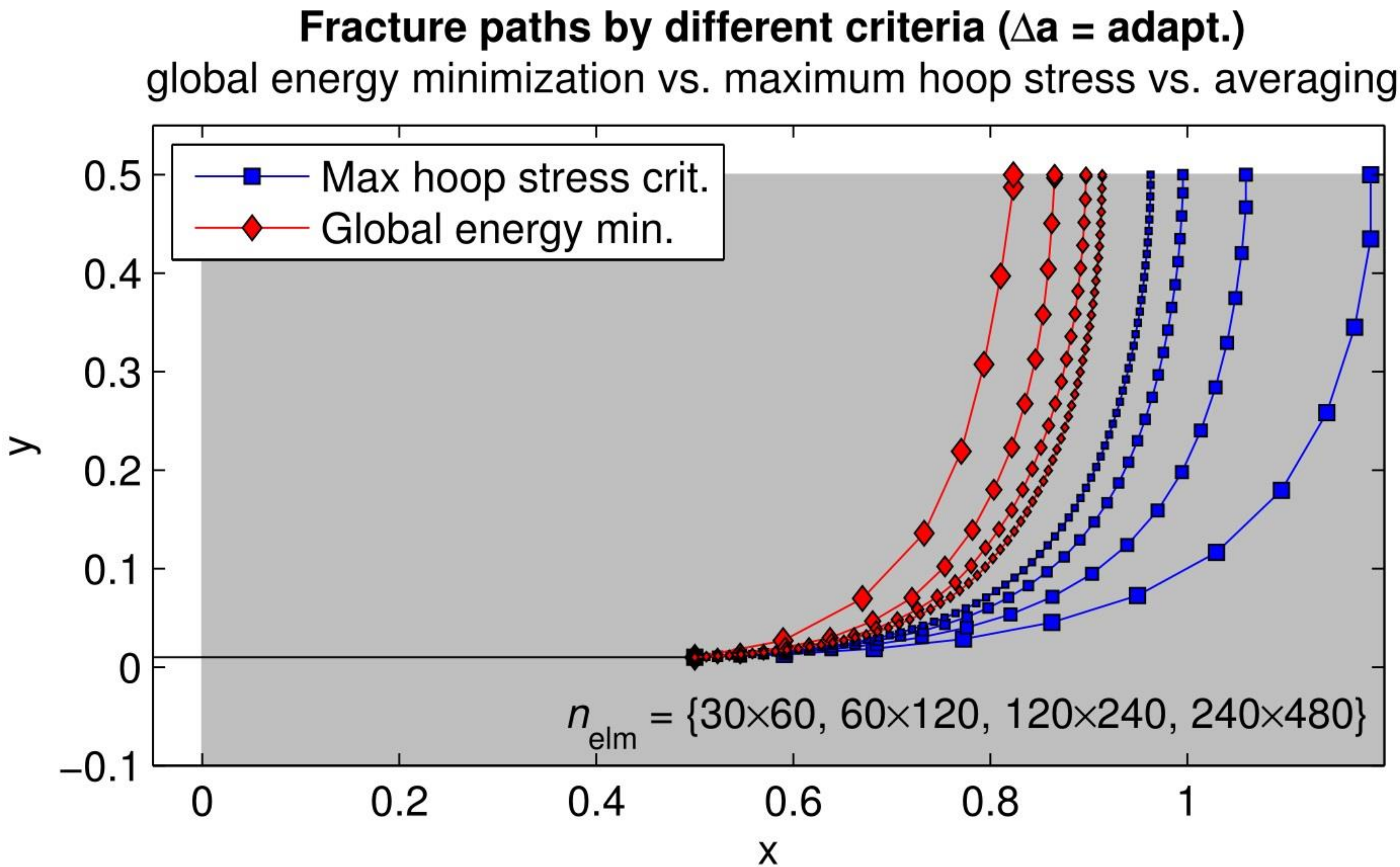
Results: Double cantilever problem



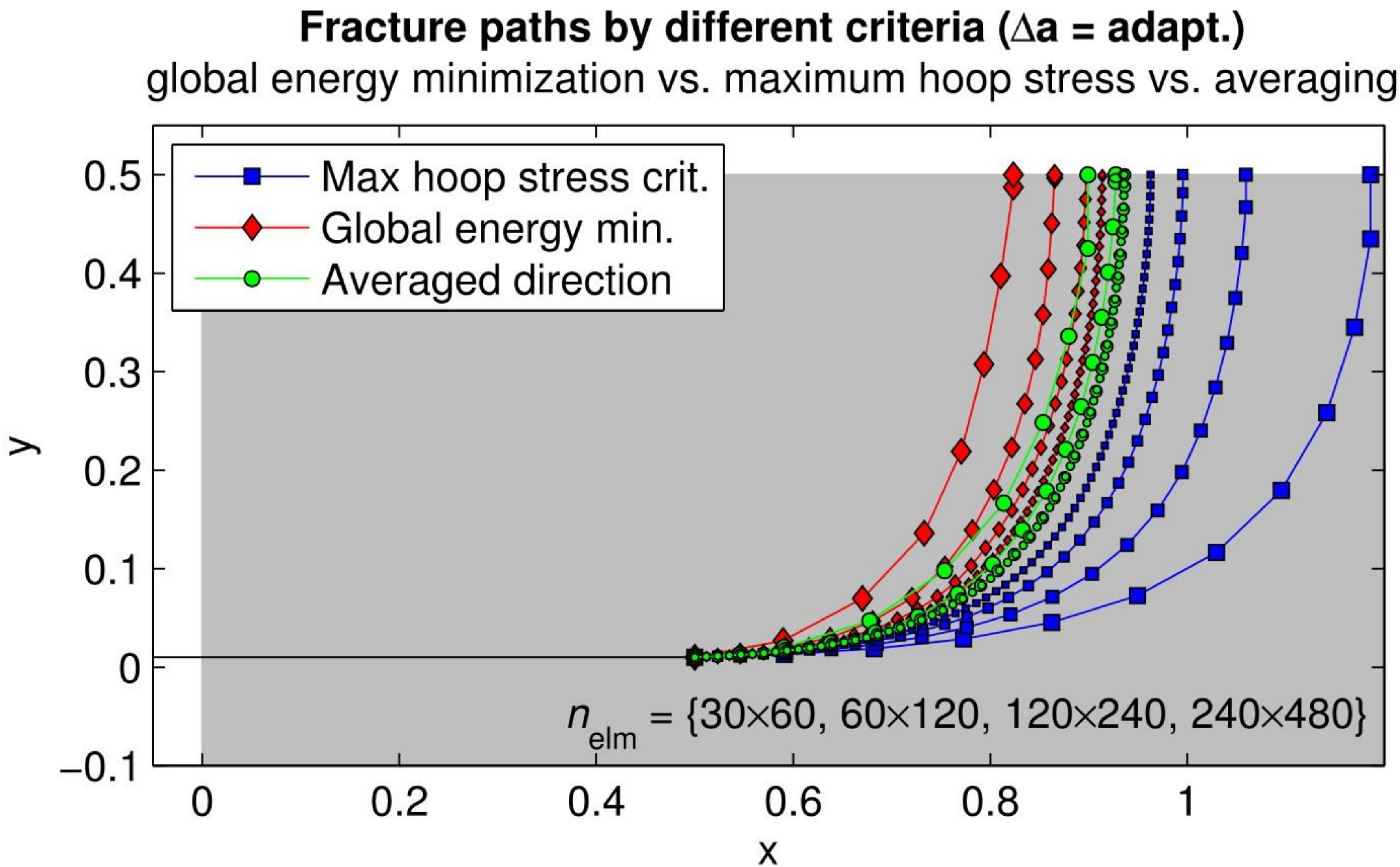
Results: Double cantilever problem



Results: Double cantilever problem



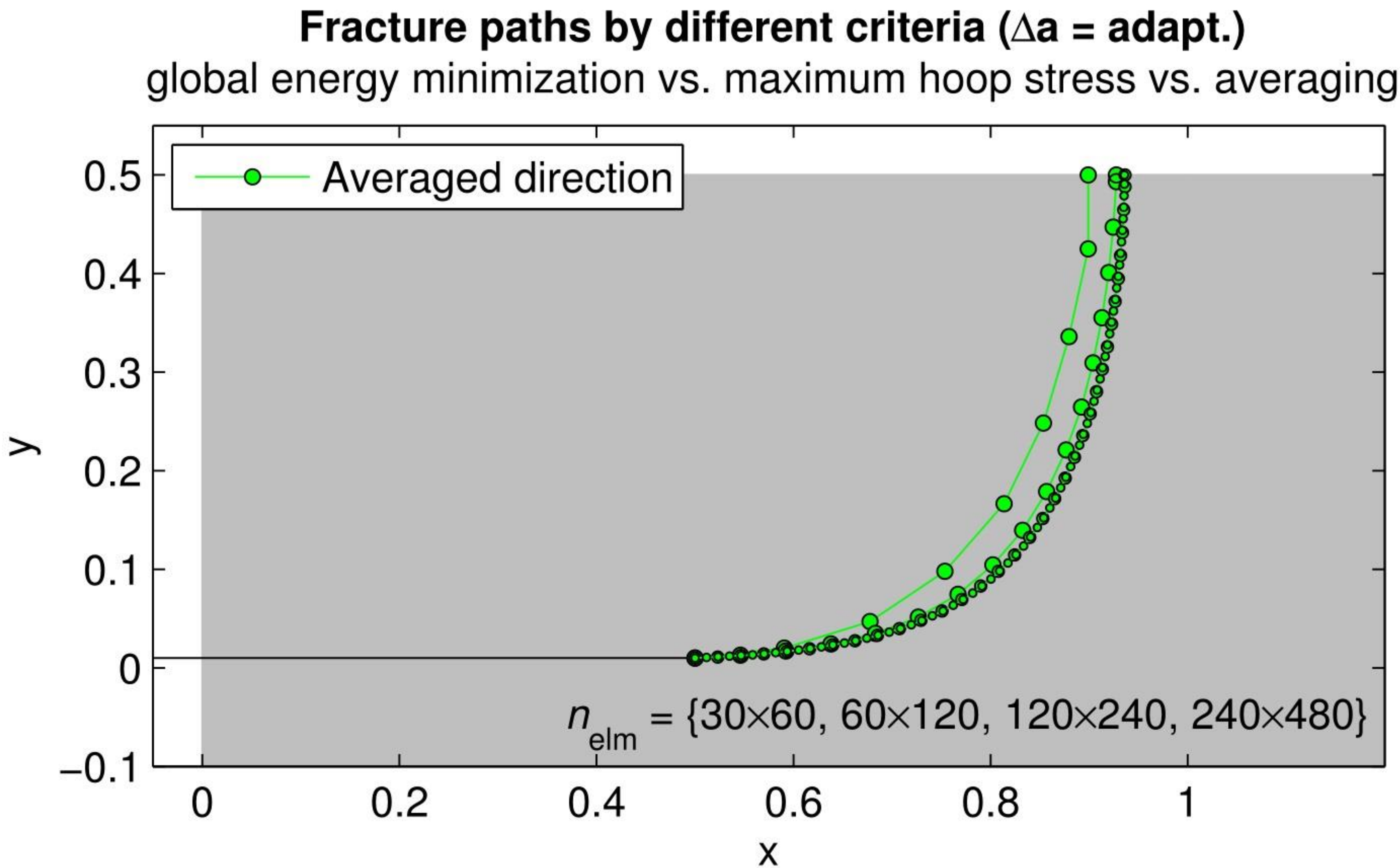
Results: Double cantilever problem



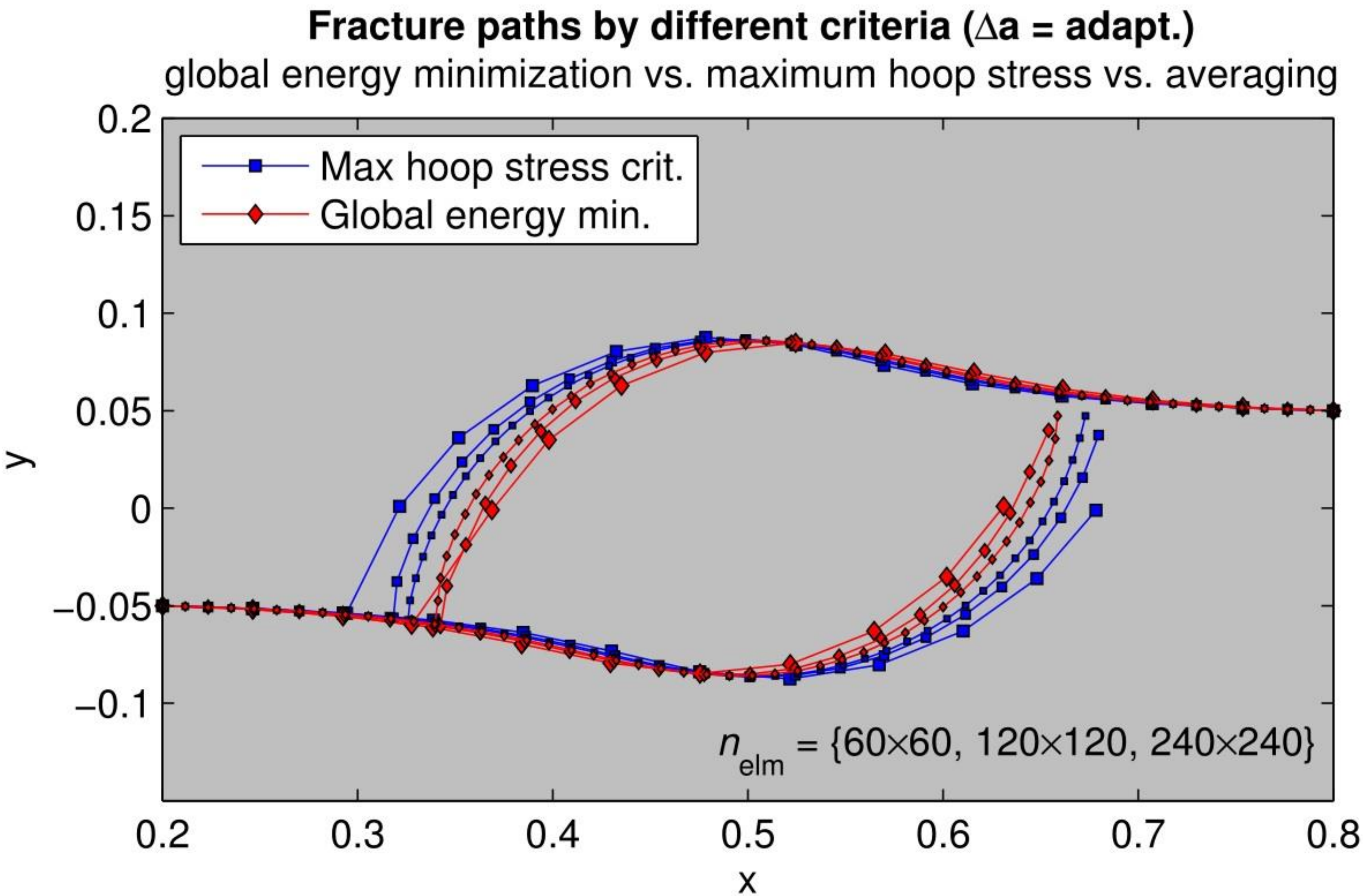
CARDIFF UNIVERSITY PRIFYSGOL CAERDYDD | MAM Institute of Mechanics & Advanced Materials

Results: Double cantilever problem

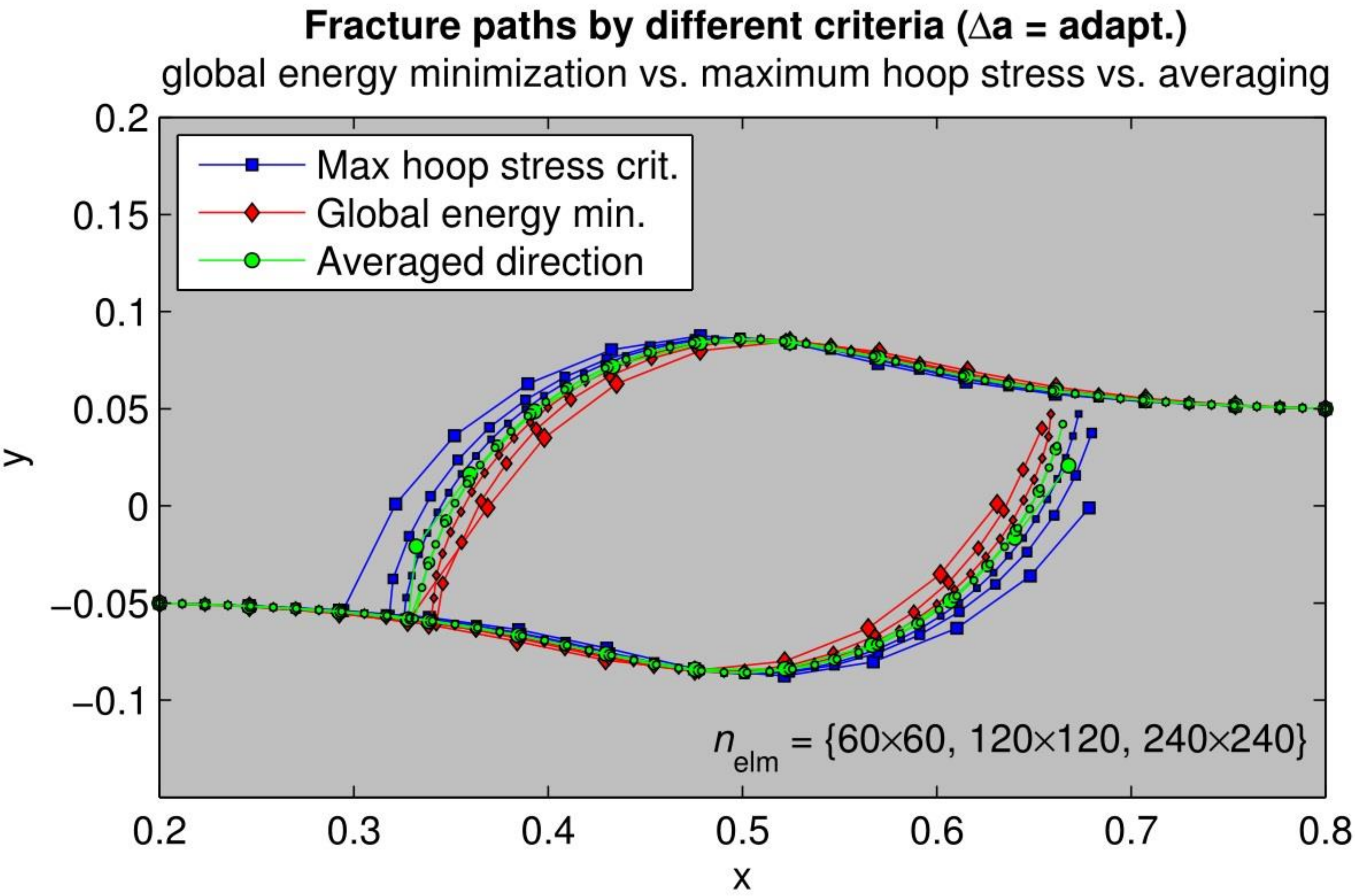
Soitec



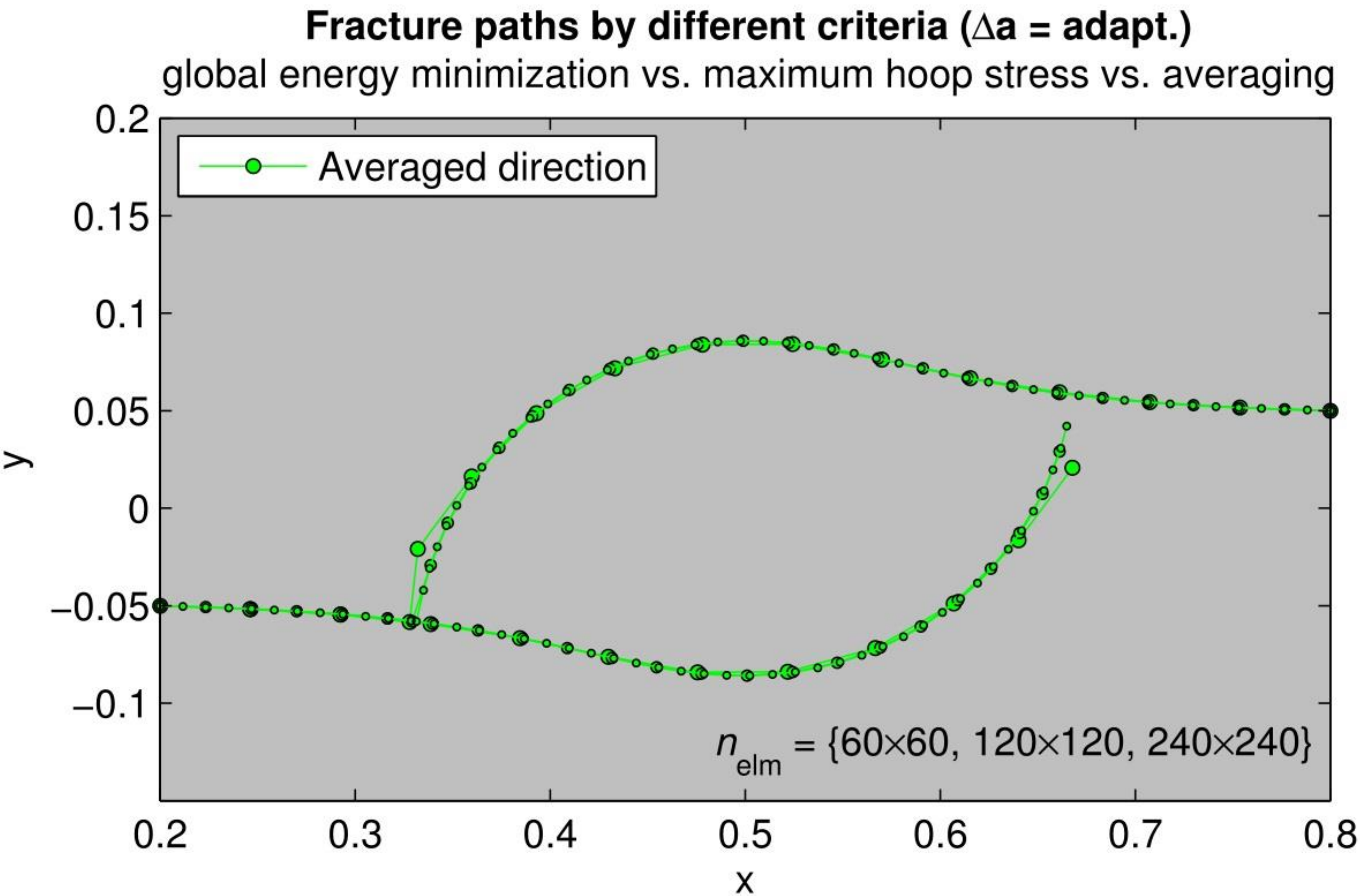
Results: Two Edge Cracks



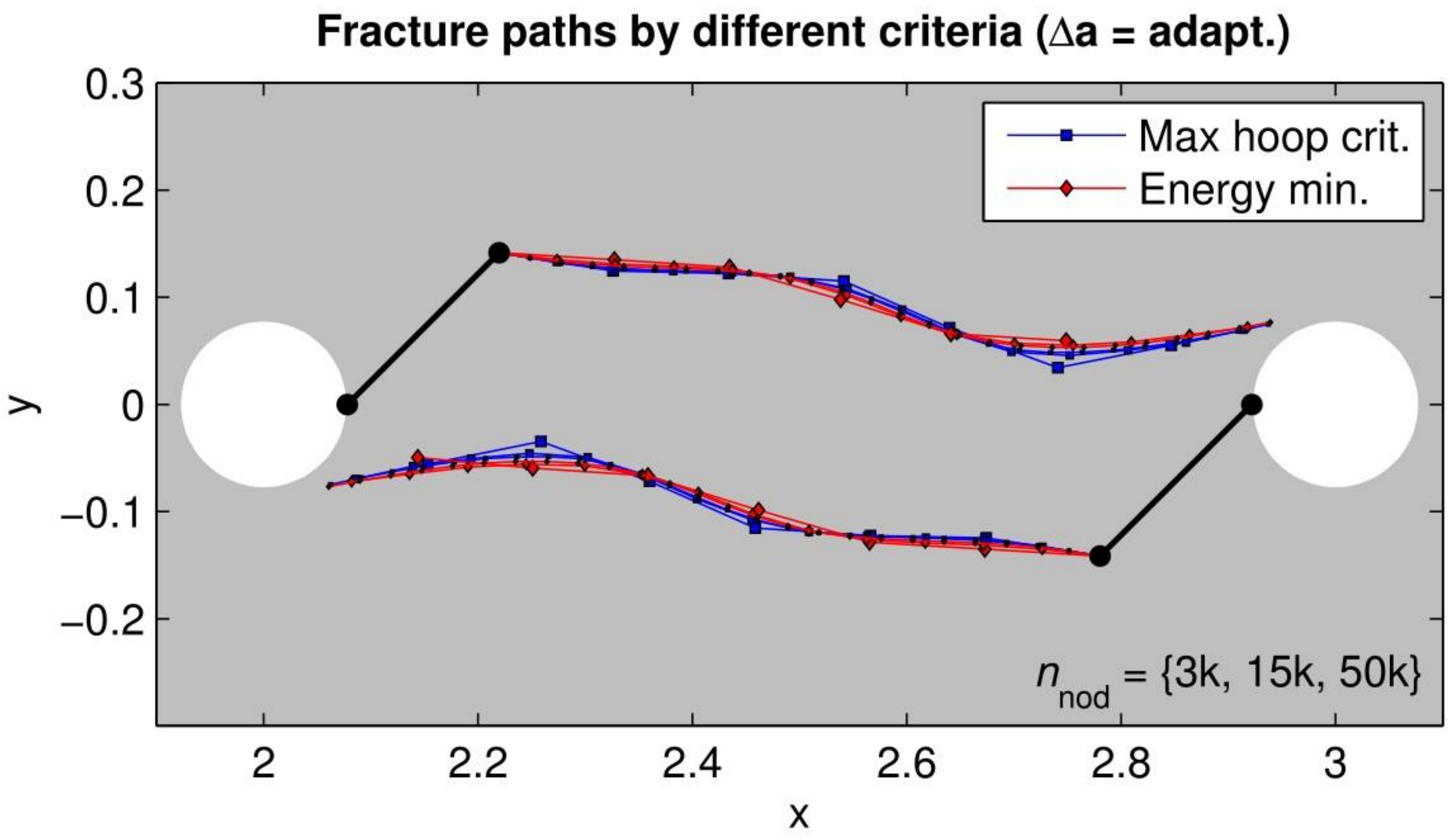
Results: Two Edge Cracks



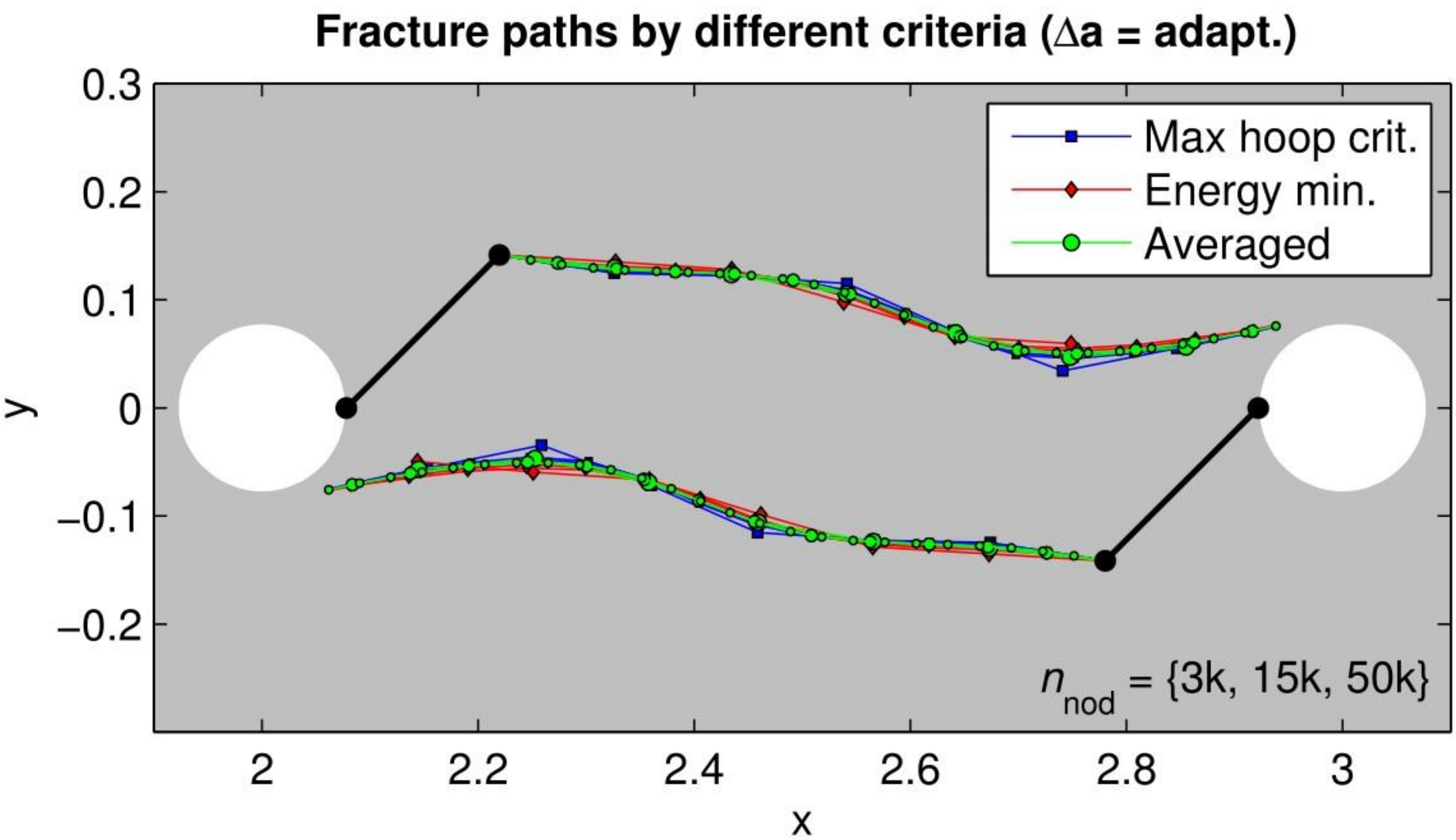
Results: Two Edge Cracks



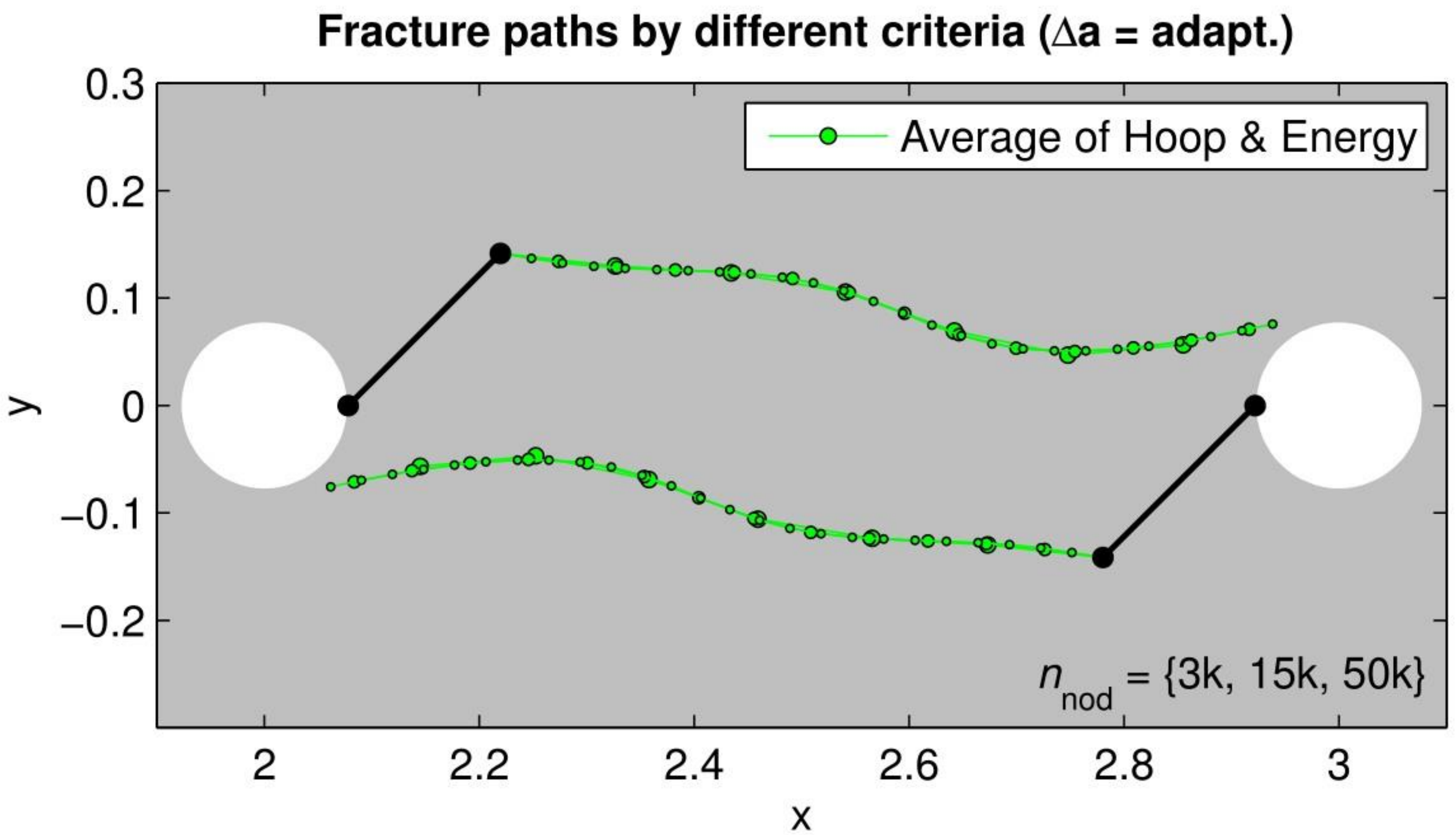
Results: Two cracks protruding from holes



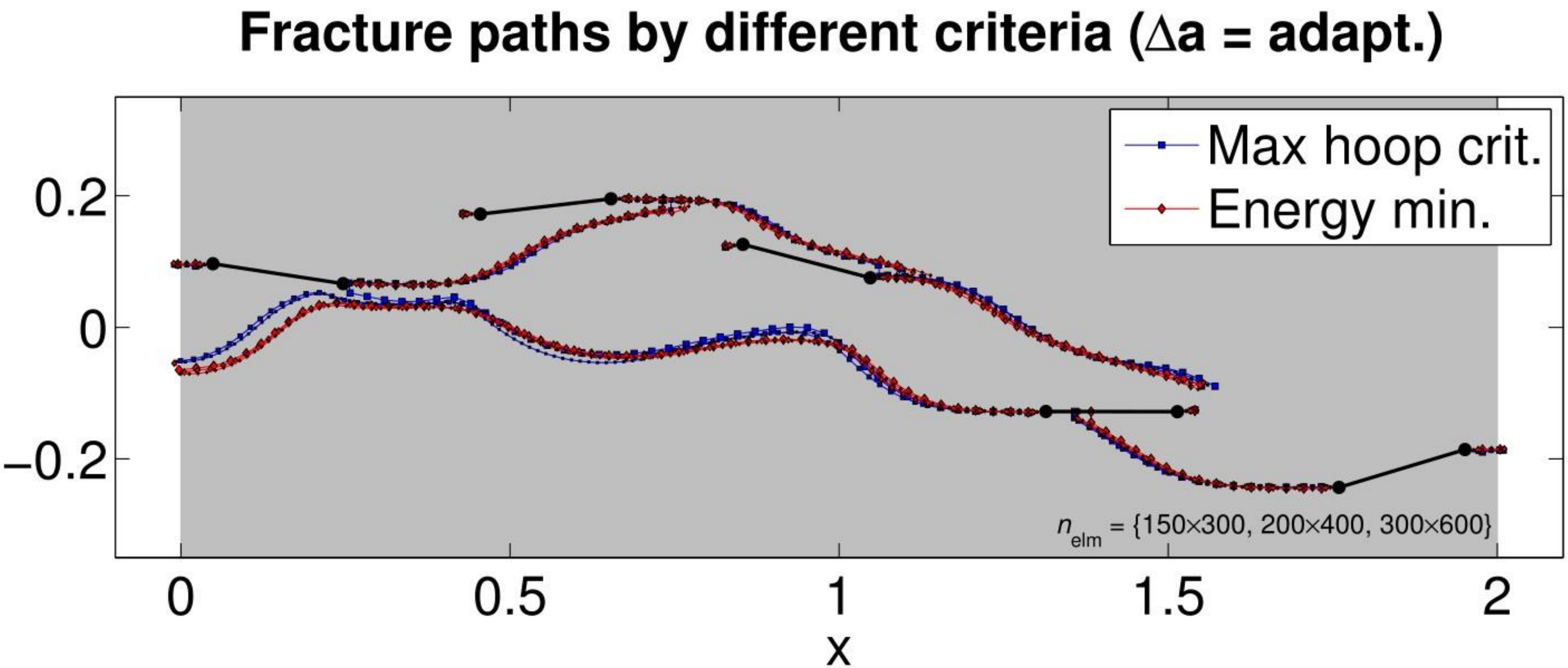
Results: Two cracks protruding from holes



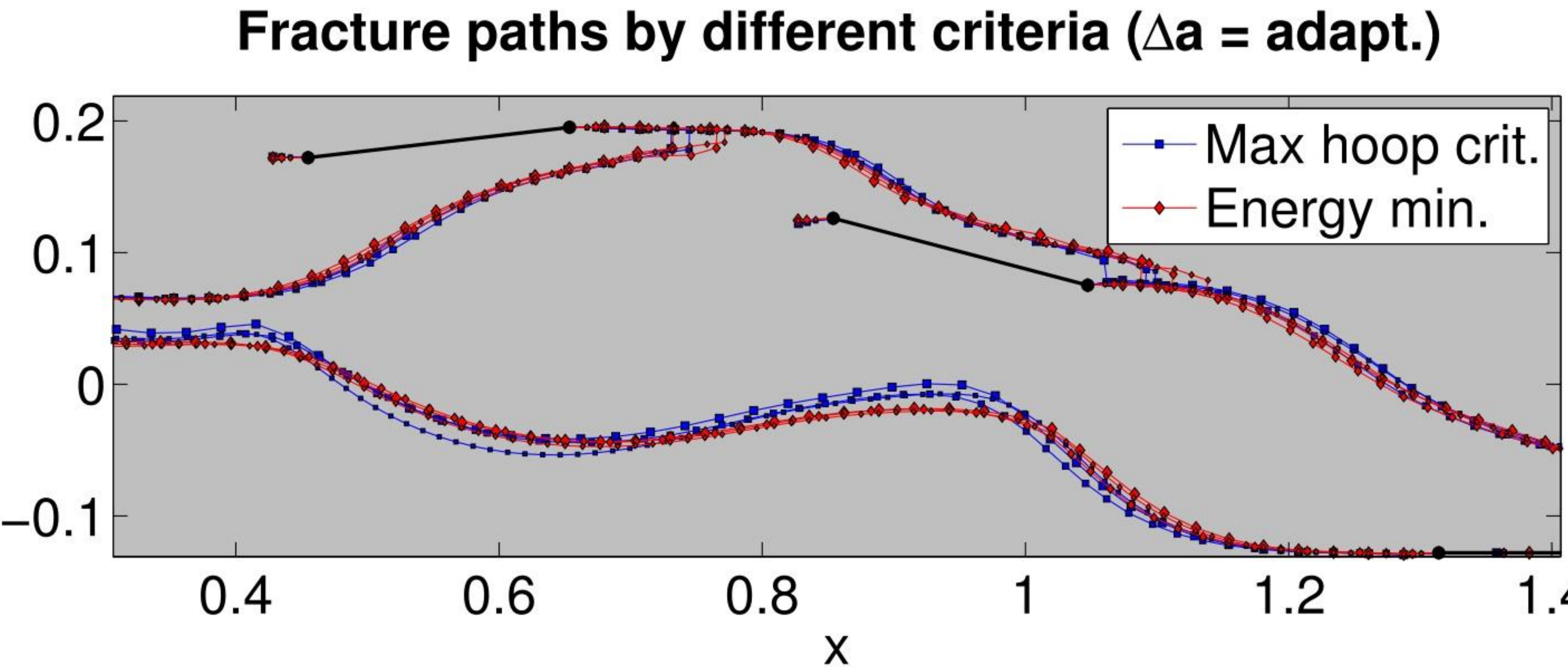
Results: Two cracks protruding from holes



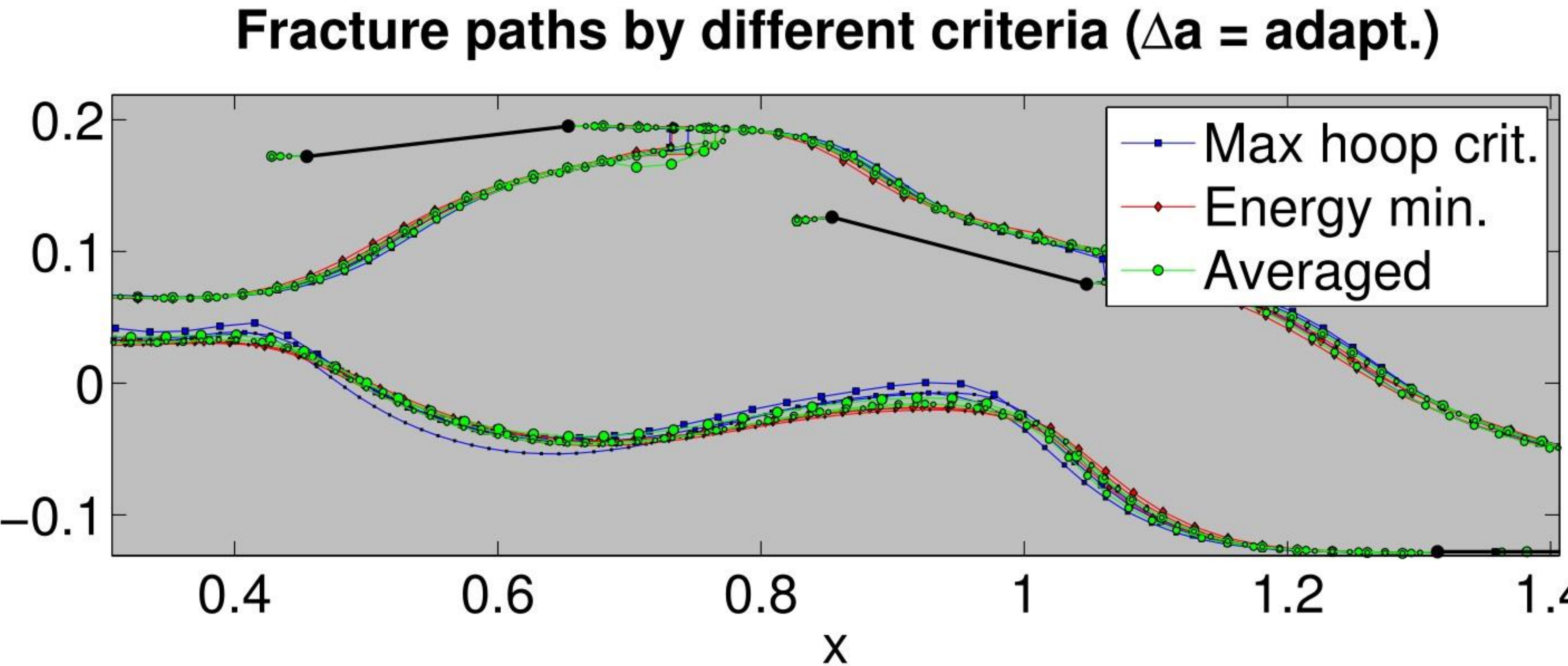
Results: 5 randomly distributed cracks



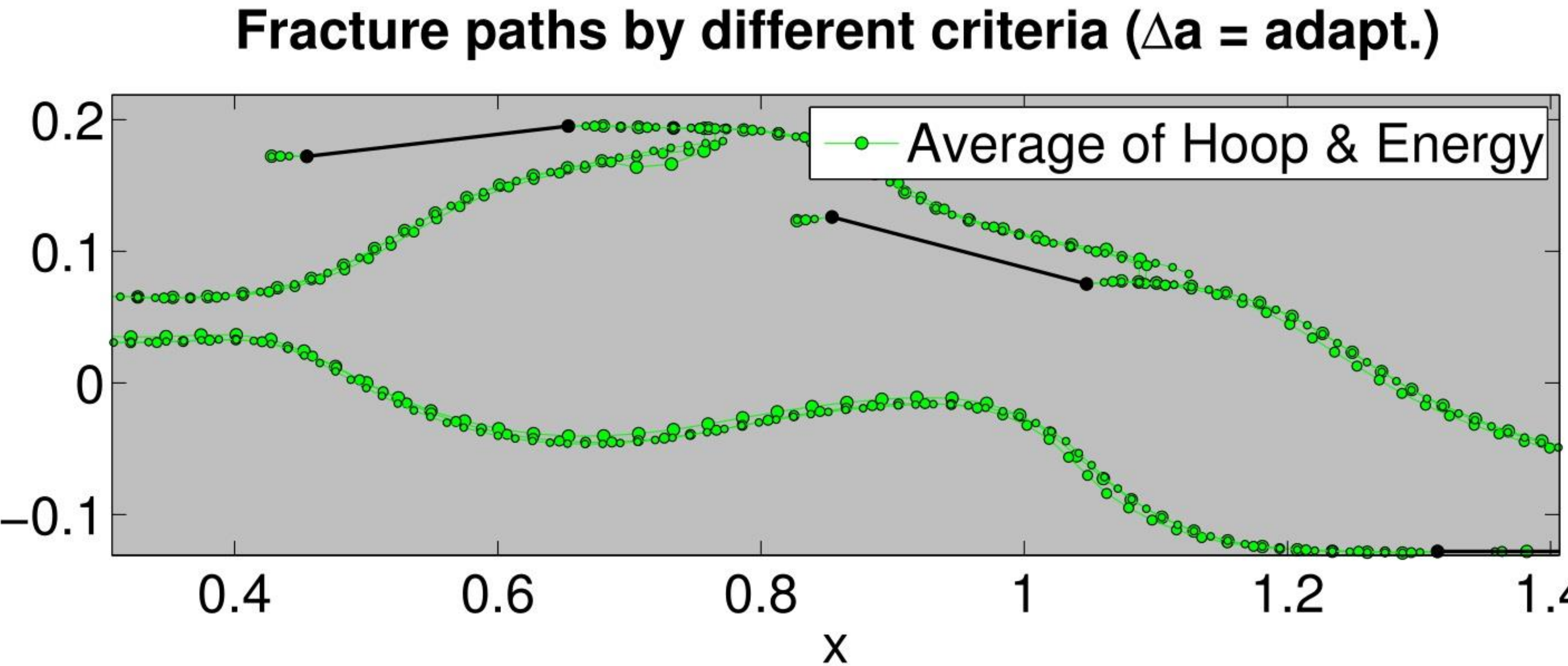
Results: 5 randomly distributed cracks



Results: 5 randomly distributed cracks



Results: 5 randomly distributed cracks



What's next

- Energy release rates w.r.t crack increment length:

$$Gs_i = -\frac{\partial \Pi}{\partial a_i} \quad , \quad Hs_{ij} = -\frac{\partial^2 \Pi}{\partial a_i \partial a_j}$$

✓ Robust crack growth criterion for multiple fractures

✓ Global energy minimization w.r.t. both θ_i , Δa_i